A Theta lift representation for the Kawazumi-Zhang and Faltings invariants of genus-two Riemann surfaces

By *Boris Pioline* , Geneva

Abstract. The Kawazumi-Zhang invariant φ for compact genus-two Riemann surfaces was recently shown to be an eigenmode of the Laplacian on the Siegel upper half-plane, away from the separating degeneration divisor. Using this fact and the known behavior of φ in the non-separating degeneration limit, it is shown that φ is equal to the Theta lift of the unique (up to normalization) weak Jacobi form of weight -2. This identification provides the complete Fourier-Jacobi expansion of φ near the non-separating node, gives full control on the asymptotics of φ in the various degeneration limits, and provides an efficient numerical procedure to evaluate φ to arbitrary accuracy. It also reveals a mock-type holomorphic Siegel modular form of weight -2 underlying φ. From the general relation between the Faltings invariant, the Kawazumi-Zhang invariant and the discriminant for hyperelliptic Riemann surfaces, a Theta lift representation for the Faltings invariant in genus two readily follows.

1. Introduction

The Kawazumi-Zhang invariant, introduced in [25,32], is a real-valued function $\varphi(\Sigma)$ on the moduli space $\mathcal{M}_h$ of compact Riemann surfaces Σ of genus $h \geq 1$. One way of defining it is through the spectrum of the Laplacian Δ_Σ with respect to the Arakelov metric on Σ,

$$\varphi(\Sigma) = \sum_{\ell>0} \frac{2}{\lambda_\ell} \sum_{m,n=1}^{h} \left| \int_\Sigma \phi_\ell \, \omega_m \, \bar{\omega}_n \right|^2 \tag{1}$$

where $(\omega_1, \dots, \omega_h)$ is an orthonormal basis of holomorphic differentials on Σ, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of Δ_Σ, and ϕ_ℓ a corresponding orthonormal basis of real square-integrable eigenmodes. For genus 2, $\varphi(\Sigma) \equiv \varphi(\Omega)$ is a function of the period matrix Ω, and defines a real-analytic modular function on the Siegel upper half-plane $\mathcal{H}_2$, away from the separating degeneration divisor. The Kawazumi-Zhang invariant is a close cousin [11] of the Faltings invariant $\delta_F(\Sigma)$ [17], which plays an important role in arithmetic geometry. Its asymptotic behavior near the boundaries of the moduli space $\mathcal{M}_h$ in arbitrary genus was investigated in [10, 12, 30].

While Faltings' invariant made an appearance in studies of bosonisation in conformal field theory [1], the genus-two Kawazumi-Zhang invariant has entered the physics literature in a recent analysis of the low energy expansion of the two-loop four-graviton amplitude in superstring theory [13]: the leading $D^4\mathcal{R}^4$ interaction is proportional to the Weil-Petersson volume of the moduli space $\mathcal{M}_2$, while the next-to-leading $D^6\mathcal{R}^4$ interaction is proportional to the integral of φ times the same Weil-Petersson volume form on $\mathcal{M}_2$. With hindsight from various physics conjectures, it was proven in [14] that φ is an eigenmode of the Laplacian $\Delta_{Sp(4)}$

on the Siegel upper half-plane, up to a source term supported on the separating degeneration divisor,

$$\left[\Delta_{Sp(4)} - 5\right] \varphi = -2\pi \, \det(\mathrm{Im}\,\Omega)\, \delta^{(2)}(v) \,, \tag{2}$$

where v is the off-diagonal element in the period matrix

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} = \begin{pmatrix} \rho_1 & v_1 \\ v_1 & \sigma_1 \end{pmatrix} + \mathrm{i} \begin{pmatrix} \rho_2 & v_2 \\ v_2 & \sigma_2 \end{pmatrix} \,. \tag{3}$$

As we shall see, this partial differential equation provides strong constraints on the asymptotic behavior at the boundaries of $\mathcal{M}_2$.

In the maximal degeneration limit, where all entries in the imaginary part Ω_2 of the period matrix $\Omega = \Omega_1 + \mathrm{i}\Omega_2$ are scaled to infinity, it was shown in [10, 14] that

$$\varphi(\Omega) = \frac{\pi}{6}\left[L_1 + L_2 + L_3 - \frac{5\, L_1 L_2 L_3}{L_1 L_2 + L_2 L_3 + L_3 L_1}\right] + \mathcal{O}(1/L_i^2) \tag{4}$$

where $0 < L_3 \leq L_1 \leq L_2$ parametrize the imaginary part of Ω in the standard fundamental domain of the action of $GL(2,\mathbb{Z})$ on the space of 2×2 positive definite real matrices,

$$\Omega_2 = \begin{pmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{pmatrix} \,. \tag{5}$$

This parametrization is motivated by the connection to two-loop supergravity amplitudes, where the L_i's play the role of Schwinger time parameters [14, 19, 20]. The leading term in (4) is an exact solution of (2) with no source term, which was one of the hints towards the differential equation (2) in [14].

In the minimal degeneration limit $\sigma \to \mathrm{i}\infty$, keeping the other entries of Ω fixed, one has instead [12, 30]

$$\varphi(\Omega) = \frac{\pi}{6} t + \varphi_0(\rho, u_1, u_2) + \mathcal{O}(1/t) \tag{6}$$

where

$$\varphi_0(\rho, u_1, u_2) = -\log\left[e^{-\pi\rho_2 u_2^2} \left| \frac{\theta(\rho, \rho u_2 - u_1)}{\eta(\rho)} \right| \right] \,. \tag{7}$$

Here, $v = \rho u_2 - u_1$ where u_1, u_2 are real, $t = \sigma_2 - u_2^2 \rho_2$ is non-negative, $\eta(\rho)$ is Dedekind's eta function and $\theta(\rho, v) = \sum_{n\in\mathbb{Z}} e^{\mathrm{i}\pi(n+\frac12)^2\rho + 2\pi\mathrm{i}(n+\frac12)(v+\frac12)}$ is Jacobi's theta series. The first two terms in (6) satisfy (2) up to terms of order $1/t$. Up to the order displayed, each term in the Laurent expansion around $t = \infty$ is a real-analytic function of ρ, u_1, u_2 invariant under the Jacobi subgroup $\Gamma_J = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \ltimes \mathbb{Z}$ of the Siegel modular group $\Gamma = Sp(4,\mathbb{Z})$ (i. e. a Jacobi form of weight zero and index zero). As we shall see, this structure extends to all orders in $1/t$.

Finally, in the separating degeneration limit $v \to 0$, keeping ρ, σ fixed, one has [12, 30]

$$\varphi(\Omega) = -\log\left|2\pi v\, \eta^2(\rho)\eta^2(\sigma)\right| + \mathcal{O}(|v|^2 \log|v|) \,. \tag{8}$$

Eq. (8) is consistent with the differential equation (2) at the order stated, with the logarithmic behavior at $v = 0$ being responsible for the delta-function source term. In [14] it was shown using the properties above that the average value of φ on $\mathcal{M}_2$ with respect to the Weil-Petersson volume form is equal to 3/2, verifying a prediction from S-duality in superstring theory.

Our goal in this work is to determine the complete asymptotics of the invariant $\varphi(\Omega)$ in the degeneration limits (4) and (6), and more generally, obtain the complete Fourier expansion with respect to Ω_1. To this aim, in §3.2 we shall construct a real-analytic Siegel modular form $\tilde{\varphi}(\Omega)$ on $\mathcal{M}_2$ that satisfies (2), (4) and (6). Since $\varphi - \tilde{\varphi}$ is square integrable on $\mathcal{M}_2$ and an eigenmode of $\Delta_{Sp(4)}$ with non-negative eigenvalue, it must therefore vanish (Theorem 1 in §3.3). $\tilde{\varphi}$ is constructed as the Theta lift of the unique weight -2 weak Jacobi form θ^2/η^6 (see Eq. (41)). This parallels the construction of the log-norm of the Igusa cusp form Ψ_{10} as the Theta lift of the unique weight 0 weak Jacobi form (also known as the elliptic genus of K3) due to Kawai [23], which we review in §3.1. Since singular Theta lifts were studied extensively in [7, 9], we refer to these works for issues of convergence. For the convenience of the reader however, we shall rederive the Fourier expansions at a physicist's level of rigor. Using the relation between the Faltings invariant, the Kawazumi-Zhang invariant and Ψ_{10} established in [11], a Theta lift representation for the Faltings invariant is readily obtained (Corollary 3 in §3.3).

The Theta lift representation of φ has several interesting consequences, considered in §3.3 and §4. First, it gives complete control over the asymptotics in the various degeneration limits, and provides an efficient numerical procedure to evaluate φ to arbitrary accuracy. This is likely to have useful applications in Arakelov geometry. Second, it reveals a 'holomorphic prepotential' $F_1(\Omega)$ which generates φ through the action of (the real part of) the Siegel-Maass raising operator (Eq. (83)). F_1 transforms non-homogeneously under $Sp(4, \mathbb{Z})$, giving an explicit example of a mock-type Siegel modular form. Third, it implies that φ is an eigenmode of an invariant quartic differential operator (Eq. (91)). It would be interesting to understand whether the differential equations (2) and (91) can be generalized to higher genus.

From the physics point of view, the results obtained here will be key for checking S-duality predictions for $D^6\mathcal{R}^4$ couplings in string theory [29]. In a different vein, it is worth noting that the same type of prepotential F_1 appears in the physics literature when computing one-loop corrections to the holomorphic prepotential in heterotic vacua with $\mathcal{N} = 2$ supersymmetry [5, 22, 28]. In that context, F_1 encodes a subset of the Gromov-Witten invariants in the dual type IIA string theory compactified on a suitable K3-fibered Calabi-Yau threefold. This analogy suggests that the product of the moduli space of genus-two Riemann surfaces times the Poincaré upper half-plane $\mathcal{H}_1$ (parametrizing the size s of the base of the K3-fibration) may carry some canonical special Kähler metric derived from a prepotential $F(s, \rho, v, \sigma) = s(\rho\sigma - v^2) + F_1 + \mathcal{O}(e^{-s})$, where F_1 is the holomorphic prepotential underlying the Kawazumi-Zhang invariant. It would be very interesting to find a string theory compactification whose moduli space carries this putative metric, and compute the $\mathcal{O}(e^{-s})$ corrections using mirror symmetry techniques.

2. Refined degeneration formulae

In this section, we shall attempt to improve the accuracy of the asymptotic expansions (4), (6) and (8) by requiring consistency with the Laplace equation (2) and invariance under

the Jacobi group Γ_J. This section is heuristic, and the proof that φ actually satisfies these improved asymptotic expansions is deferred to §3. This attempt is inspired by a study of two-loop amplitudes in superstring theory [29], and in turn, in combination with insights gained from a study of generalized Borcherds lifts [2–4], inspired the educated guess considered in §3.2. The reader uninterested by the source of this guess can safely skip to §3.

Starting with the minimal non-separating degeneration, we observe that the expansion (6) can be strengthened, consistently with the Laplace equation (2) to exponential accuracy, to

$$\varphi(\Omega) = \frac{\pi}{6}t + \varphi_0 + \frac{\varphi_1}{t} + \mathcal{O}(e^{-t}) \,, \tag{9}$$

where φ_1 is a function of ρ, u_1, u_2 to be determined. Indeed, decomposing the Laplace operator into

$$\Delta_{Sp(4)} = \Delta_t + \Delta_\rho + t\,\Delta_u + \Delta_{\sigma_1} \tag{10}$$

where

$$\begin{aligned} \Delta_t =& t^2\partial_t^2 - t\partial_t \,, \quad \Delta_\rho = \rho_2^2(\partial_{\rho_1}^2 + \partial_{\rho_2}^2) \,, \quad \Delta_u = \frac{1}{2\rho_2}|\rho\,\partial_{u_1} + \partial_{u_2}|^2 \,, \\ \Delta_{\sigma_1} =& -(t+\rho_2 u_2^2)\partial_{\sigma_1}^2 + (2t\rho_2 u_2\partial_{u_1} - 2\rho_2^2 u_2^2\partial_{\rho_1})\partial_{\sigma_1} \,, \end{aligned} \tag{11}$$

and using the fact that φ_0, defined in (7), satisfies

$$\Delta_\rho\,\varphi_0 = 0 \,, \quad \Delta_u\,\varphi_0 = \pi \,, \tag{12}$$

we see that the Laplace equation (2) is satisfied at order $\mathcal{O}(e^{-t})$ provided φ_1 satisfies

$$\Delta_\rho\,\varphi_1 = 2\varphi_1 \,, \quad \Delta_u\,\varphi_1 = 5\varphi_0 \,. \tag{13}$$

Invariance of φ under Γ requires that φ_1 be a real-analytic Jacobi form of zero weight and zero index. On the other hand, the maximal degeneration limit (4) requires that, in the limit $\rho_2 \to \infty$ keeping ρ_1, u_1, u_2 fixed (with $u_2 \in [0,1]$),

$$\varphi_0 \sim \frac{\pi}{6}\rho_2(1-6u_2+6u_2^2) \,, \quad \varphi_1 \sim \frac{5\pi}{6}\rho_2^2 u_2^2(u_2-1)^2 \,. \tag{14}$$

The first equation is of course satisfied by (7). It is suggestive to rewrite these limits in terms of the Bernoulli polynomials $B_2(x) = x^2 - x + \frac{1}{6}$, $B_4(x) = x^2(x-1)^2 - \frac{1}{30}$:

$$\varphi_0 \sim \pi\rho_2 B_2(u_2) \,, \quad \varphi_1 \sim \frac{5\pi}{6}\rho_2^2\left(B_4(u_2) + \tfrac{1}{30}\right) \tag{15}$$

A solution to (13) obeying these boundary conditions can be obtained as a linear combination

$$\varphi_1 = \frac{5}{16\pi^2\rho_2}\,\mathcal{D}_{2,2}(\rho;v) + \frac{5}{2\pi}\,E^\star(2;\rho) \tag{16}$$

of the standard non-holomorphic Eisenstein series

$$E^\star(s;\rho) = \frac{1}{2}\pi^{-s}\Gamma(s)\sum_{(m,n)\neq(0,0)}\left[\frac{\rho_2}{|m\rho+n|^2}\right]^s \tag{17}$$

and the Kronecker-Eisenstein series introduced in [31]

$$\mathcal{D}_{a,b}(\rho;v) \equiv \frac{(2\mathrm{i}\rho_2)^{a+b-1}}{2\pi\mathrm{i}} \sum_{(m,n)\neq(0,0)} \frac{e^{2\pi\mathrm{i}(n\,u_2+m\,u_1)}}{(m\rho+n)^a(m\bar\rho+n)^b}\ , \tag{18}$$

where a, b are non-negative integers. $\mathcal{D}_{a,b}(\rho;v)$ is a real-analytic Jacobi modular form of weight $(1-b, 1-a)$ and zero index, with Fourier expansion

$$\mathcal{D}_{a,b}(\rho;v) = \sum_{m=0}^{\infty} D_{a,b}(q^m\,x) + (-1)^{a+b} \sum_{m=1}^{\infty} D_{a,b}(q^m\,x^{-1}) + \frac{(4\pi\rho_2)^{a+b-1}}{(a+b)!}\, B_{a+b}\,(u_2)\ , \tag{19}$$

where $x = e^{2\pi\mathrm{i}v} = e^{2\pi\mathrm{i}(u_2\rho-u_1)}$, $q = e^{2i\pi\rho}$, $B_\alpha(x)$ are the Bernoulli polynomials, and $D_{a,b}(x)$ are the Bloch-Wigner-Ramakrishnan single-valued polylogarithms [31],

$$\begin{aligned} D_{a,b}(x) =&(-1)^{a-1} \sum_{k=a}^{a+b-1} 2^{a+b-1-k} \binom{k-1}{a-1} \frac{(-\log|x|)^{a+b-1-k}}{(a+b-1-k)!}\,\mathrm{Li}_k(x) \\ &+ (-1)^{b-1} \sum_{k=b}^{a+b-1} 2^{a+b-1-k} \binom{k-1}{b-1} \frac{(-\log|x|)^{a+b-1-k}}{(a+b-1-k)!}\,\overline{\mathrm{Li}_k(x)}\ . \end{aligned} \tag{20}$$

It is easy to check that the differential equations (13) are obeyed, by checking the action on the seed of the Poincaré series (i.e. setting $m = 0, n = 1$) and using the second Kronecker limit formula, which states

$$\varphi_0 = \frac{1}{2}\,\mathcal{D}_{1,1}(\rho;v)\ . \tag{21}$$

Moreover, the equality (16) predicts an additional subleading term in (15),

$$\varphi_1 = \frac{5\pi}{6}\rho_2^2 B_4(u_2) + \frac{\pi}{36}\,\rho_2^2 + \frac{5\zeta(3)}{4\pi^2}\,\rho_2^{-1} + \mathcal{O}(e^{-2\pi\rho_2}) \tag{22}$$

The third term in (22) requires a subleading correction to the maximal degeneration limit (4),

$$\varphi(\Omega) = \frac{\pi}{6}\left[L_1+L_2+L_3 - \frac{5\,L_1L_2L_3}{L_1L_2+L_2L_3+L_3L_1}\right] + \frac{5\zeta(3)}{4\pi^2(L_1L_2+L_2L_3+L_3L_1)} + \dots \tag{23}$$

The additional term is an exact solution of the Laplace equation (2).

With the hindsight gained from a study of generalized Borcherds lifts [2], the estimates (9) and (23), if true, strongly suggest that $\varphi(\Omega)$ is the Theta lift of an almost, weakly holomorphic Jacobi form of weight $-1/2$ and depth 1, motivating the educated guess in §3.2. We shall prove in §3.3 that this estimates do in fact hold with exponential accuracy.

3. The Kawazumi-Zhang invariant as a Theta lift

In this section, using a suitable Theta lift, we construct a real-analytic Siegel modular form $\tilde\varphi$ that satisfies the differential equation (2) and asymptotic behaviors (4), (6), (8) in the various degeneration limits – hence must coincide with φ. As a warm-up, we start by recalling the Theta lift representation of the log-norm of the discriminant of genus two Riemann surfaces, following [23].

3.1. The Igusa cusp form Ψ_{10} as a Theta lift. Recall that the log-norm $\log ||\Psi_{10}|| = \log[(\det\Omega_2)^5|\Psi_{10}|]$ of the weight 10 Igusa cusp form (normalized as 2^{-12} times the product of the squares of the ten Thetanullwerte) can be represented as a regularized modular integral[1)]

$$\begin{aligned} \log ||\Psi_{10}||(\Omega) = &- \frac{1}{4}\int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2}\,\left[\Gamma_{3,2}^{\rm even}(\Omega;\tau)\,h_0(\tau) + \Gamma_{3,2}^{\rm odd}(\Omega;\tau)\,h_1(\tau) - 20\,\tau_2\right] \\ &- 5\log\left(\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}\right)\,, \end{aligned} \tag{24}$$

where $\mathcal{F}_1 = \{\tau\in\mathcal{H}_1, |\tau|>1, -\frac12<\tau_1\le\frac12\}$ is the standard fundamental domain for the action of $SL(2,\mathbb{Z})$ on $\mathcal{H}_1$, $\mathrm{d}^2\tau = \mathrm{d}\tau_1\mathrm{d}\tau_2$, γ_E is the Euler-Mascheroni constant, and $\Gamma_{3,2}^{\rm even|odd}$ are partition functions for even (shifted) lattices of signature (3,2), or Siegel-Narain theta series,

$$\begin{aligned} \Gamma_{3,2}^{\rm even|odd}(\Omega;\tau) =&\tau_2 \sum_{\substack{(m_1,m_2,n^1,n^2)\in\mathbb{Z}^4\\ b\in 2\mathbb{Z}|2\mathbb{Z}+1}} q^{\frac14 p_L^2}\,\bar q^{\frac14 p_R^2} \\ p_{\rm R}^2 =& \frac{|m_2-\rho\,m_1+\sigma n^1+(\rho\sigma-v^2)\,n^2-b\,v|^2}{\rho_2\,\sigma_2-v_2^2}, \\ p_{\rm L}^2 =& p_R^2+4m_i n^i+b^2\,. \end{aligned} \tag{25}$$

For $v\to0$, $\Gamma_{3,2}^{\rm even}\to\Gamma_{2,2}\,\theta_3(2\tau), \Gamma_{3,2}^{\rm odd}\to\Gamma_{2,2}\,\theta_2(2\tau)$ where $\Gamma_{2,2}$ is the partition function of the even self-dual lattice with signature $(2,2)$,

$$\Gamma_{2,2}(\rho,\sigma;\tau) = \tau_2 \sum_{(m_1,m_2,n^1,n^2)\in\mathbb{Z}^4} q^{\frac14 p_L^2}\,\bar q^{\frac14 p_R^2}|_{b=v=0}\,. \tag{26}$$

Thus, $\Gamma_{3,2}^{\rm even|odd}$ are modular forms of weight $1/2$ under $\Gamma_0(4)$. Under general modular transformations of τ,

$$\begin{aligned} \Gamma_{3,2}^{\rm even}(\tau+1) =&\Gamma_{3,2}^{\rm even}(\tau)\,,\quad \Gamma_{3,2}^{\rm odd}(\tau+1) = \mathrm{i}\,\Gamma_{3,2}^{\rm odd}(\tau) \\ \Gamma_{3,2}^{\rm even}(-1/\tau) =& \frac{1-\mathrm{i}}{2}\,\tau^{1/2}\,[\Gamma_{3,2}^{\rm even}(\tau)+\Gamma_{3,2}^{\rm odd}(\tau)]\,, \\ \Gamma_{3,2}^{\rm odd}(-1/\tau) =& \frac{1-\mathrm{i}}{2}\,\tau^{1/2}\,[\Gamma_{3,2}^{\rm even}(\tau)-\Gamma_{3,2}^{\rm odd}(\tau)]\,. \end{aligned} \tag{27}$$

The integers m_1, m_2, n^1, n^2, b can be fit into an antisymmetric traceless matrix

$$\begin{pmatrix} 0 & -m_2 & b/2 & n^1 \\ m_2 & 0 & m_1 & -b/2 \\ -b/2 & -m_1 & 0 & -n^2 \\ -n^1 & b/2 & n^2 & 0 \end{pmatrix}\,, \tag{28}$$

with Pfaffian proportional to $p_L^2-p_R^2$, which transforms by conjugation under $Sp(4,\mathbb{Z})$.This makes it clear that $Sp(4,\mathbb{Z})$ transformations preserve the parity of b. Thus, both $\Gamma_{3,2}^{\rm even|odd}(\Omega;\tau)$

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1) The formula (24) was discovered in [23] by computing threshold corrections to gauge couplings in heterotic string theory compactified on $K3\times T^2$. The variables ρ,σ,v parametrize the complex structure, Kähler class and holonomies of a $U(1)$ connection on the torus T^2, while the integers m_i, n^i correspond to the momentum and winding numbers, and b is the electric charge.

are Siegel modular functions in the variable Ω, and so then is the result of the modular integral (24). On the other hand, h_0, h_1 are the coefficients of the theta series decomposition of the elliptic genus of $K3$,

(29)
$$\begin{aligned}\chi_{K3}(\tau,z) =& h_0(\tau)\,\theta_3(2\tau,2z) + h_1(\tau)\,\theta_2(2\tau,2z)\ ,\\ h_0(\tau) =& 24\frac{\theta_3(2\tau)}{\theta_3^2(\tau)} - 2\frac{\theta_4^4(\tau)-\theta_2^4(\tau)}{\eta^6(\tau)}\theta_2(2\tau) = 20+216q+1616q^2+\dots\\ h_1(\tau) =& 24\frac{\theta_2(2\tau)}{\theta_3^2(\tau)} + 2\frac{\theta_4^4(\tau)-\theta_2^4(\tau)}{\eta^6(\tau)}\theta_3(2\tau) = q^{-1/4}(2-128q-1026q^2+\dots)\end{aligned}$$

They are modular forms of $\Gamma_0(4)$ with weight $-1/2$. In terms of the standard generators $X_2(2\tau)=E_2(2\tau)-2E_2(4\tau)$, $\theta_2^4(2\tau)$ of the ring of $\Gamma_0(4)$ modular forms of even weight,

$$\begin{aligned} h_0 =& \frac{1}{\theta_3(2\tau)\,\Delta_6}\left[\frac{3}{16}\theta_2^{12}(2\tau) - X_2(2\tau)\theta_2^8(2\tau) + \frac54[X_2(2\tau)]^2\,\theta_2^4(2\tau)\right]\\ h_1 =& \frac{1}{\theta_2(2\tau)\,\Delta_6}\left[-\frac{9}{16}\theta_2^{12}(2\tau) + X_2(2\tau)\theta_2^8(2\tau) + \frac14[X_2(2\tau)]^2\,\theta_2^4(2\tau)\right]\end{aligned} \tag{30}$$

where $\Delta_6=[\eta(2\tau)]^{12}$ is a cusp form of weight 6. Under general modular transformations,

$$\begin{aligned} h_0(\tau+1) =& h_0(\tau)\ ,\quad h_1(\tau+1) = -\mathrm{i}h_1(\tau)\ ,\\ h_0(-1/\tau) =& \frac{1+\mathrm{i}}{2}\,\tau^{-1/2}\,[h_0(\tau)+h_1(\tau)]\ ,\\ h_1(-1/\tau) =& \frac{1+\mathrm{i}}{2}\,\tau^{-1/2}\,[h_0(\tau)-h_1(\tau)]\ ,\end{aligned} \tag{31}$$

so that the integrand of (24) is (except for the last term in the bracket, proportional to τ_2) invariant under the full modular group $SL(2,\mathbb{Z})$. It follows from (31) (see e.g. the proof of Thm 5.4 in [16]) that

(32)
$$\begin{aligned} h(\tau) =& h_0(4\tau)+h_1(4\tau)\\ =& -\frac{1}{8\theta_3(2\tau)\Delta_6}\left(-3\theta_2^{12}(2\tau)+4\theta_2^8(2\tau)X_2(2\tau)+12\theta_2^4(2\tau)X_2^2(2\tau)-16X_2^3(2\tau)\right)\\ =& \sum_{m\geq -1} c(m)q^m = 2q^{-1}+20-128q^3+216q^4-1026q^7+\dots\end{aligned}$$

is modular form of weight $-\frac12$ under $\Gamma_0(4)$ in Kohnen's plus space (*i.e.* the m-th Fourier coefficient $c(m)$ of $h(\tau)$ vanishes unless $m=0,3 \mod 4$). $h(\tau)$ has a simple pole at $\tau=\mathrm{i}\infty$ and is regular at $\tau=0$ and $\tau=\frac12$. The constant term in $h(\tau)$ makes it necessary to subtract by hand the term proportional to τ_2 in (24), in order for the integral to converge.[2)]

Using the differential equation satisfied by the lattice partition function,

$$\left[\Delta_{Sp(4)} - 4\Delta_{SL(2),1/2} + 1\right]\,\Gamma_{3,2}^{\mathrm{even}|\mathrm{odd}} = 0\ , \tag{33}$$

[2)] Alternatively, following [7] one could truncate the integration domain to $\mathcal{F}_1^\Lambda=\mathcal{F}_1\cap\{\tau_2<\Lambda\}$, insert a Kronecker regulating factor τ_2^s in the integrand, take the limit $\Lambda\to\infty$ for fixed s with $\mathrm{Re}(s)$ sufficiently large, analytically continue in s and extract the constant term in the Laurent expansion at $s=0$. The two prescriptions can be shown to agree up to an additive constant.

where $\Delta_{SL(2),w} = 4\tau_2^2\partial_{\bar\tau}(\partial_\tau + \frac{w}{2\mathrm{i}\tau_2}) + w$ is the Laplacian acting on modular forms of weight w, one sees that $\log||\Psi_{10}||$ is a real-analytic quasi-harmonic function on the Siegel upper half-plane, up to a delta function source term supported on the separating divisor,

$$\Delta_{Sp(4)}\,\log||\Psi_{10}|| = -15 + 4\pi\,\delta^{(2)}(v)\ . \tag{34}$$

Indeed, as $v\to 0$, the integrand becomes

$$\Gamma_{2,2}\left[\theta_3(2\tau)\,h_0(\tau) + \theta_2(2\tau)\,h_1(\tau)\right] - 20\tau_2 = 24\Gamma_{2,2} - 20\tau_2 \overset{\tau_2\to\infty}{\sim} 4\tau_2\ , \tag{35}$$

which leads to a logarithmic divergence. Keeping v small but non zero, and retaining the contributions from $m_i = n^i = 0, b = \pm 1$, we have

$$\log||\Psi_{10}|| \sim -\int_1^\infty \frac{\mathrm{d}\tau_2}{\tau_2} e^{-\frac{\pi\tau_2|v|^2}{\rho_2\sigma_2 - v_2^2}} = -\Gamma\left(0,\pi z\right)\ , \tag{36}$$

where $z = \frac{|v|^2}{\rho_2\sigma_2 - v_2^2}$. Using the fact that the incomplete Gamma function $\Gamma(0,\pi z)$ behaves as $-\log(\pi z)+$ analytic as $z\to 0$, and the result from [15]

$$\int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2}(\Gamma_{2,2} - \tau_2) = -\log\left[\frac{8\pi e^{1-\gamma}}{3\sqrt{3}}\rho_2\sigma_2|\eta(\rho)\eta(\sigma)|^4\right]\ , \tag{37}$$

we find

$$\log||\Psi_{10}|| = \log|\rho_2^5\sigma_2^5 v^2\eta^{24}(\rho)\eta^{24}(\sigma)| + \mathcal{O}(|v|^2)\ , \tag{38}$$

where the omitted terms vanish analytically as $v\to 0$.

Evaluating the modular integral by the standard unfolding method [7, 22], one arrives at

$$\begin{aligned}\log||\Psi_{10}||(\Omega) = &-2\pi(\rho_2+\sigma_2-v_2) + 5\log\det\Omega_2\\ &-\mathrm{Re}\Big[\sum_{(k,\ell,b)>0} c(4k\ell-b^2)\log\Big(1-e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\Big)\Big]\ ,\end{aligned} \tag{39}$$

where $(k,\ell,b)>0$ stands for $\{(k>0,\ell\geq 0)$ or $(k\geq 0,\ell>0), b\in\mathbb{Z}\}\cup\{k=\ell=0,b>0\}$, and Ω is assumed to be such that $k\sigma_2+\ell\rho_2+bv_2>0$ for all $(k,\ell,b)>0$ [23, Eq. (20)]. Eq. (39) is consistent with the Gritsenko-Nikulin product formula [21]

$$\Psi_{10}(\Omega) = e^{2\pi\mathrm{i}(\rho+\sigma-v)}\prod_{(k,\ell,b)>0}(1-e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)})^{c(4k\ell-b^2)}\ . \tag{40}$$

3.2. An educated guess. Motivated by the heuristic considerations in §2, and in analogy with the Theta lift representation of $\log||\Psi_{10}||$ reviewed in §3.1, we consider the modular integral

$$\tilde\varphi(\Omega) = -\frac{1}{2}\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\left[\Gamma_{3,2}^{\rm even}(\Omega;\tau)\,D_\tau\tilde h_0(\tau) + \Gamma_{3,2}^{\rm odd}(\Omega;\tau)\,D_\tau\tilde h_1(\tau)\right]\ , \tag{41}$$

where $D_\tau = \frac{\mathrm{i}}{\pi}(\partial_\tau - \frac{\mathrm{i}w}{2\tau_2})$ is the raising operator, mapping modular forms of weight w to modular forms of weight $w+2$, and $(\tilde h^0,\tilde h^1)$ are weight $w=-5/2$ modular forms of $\Gamma_0(4)$,

associated to the weak Jacobi form $\tilde{\phi} = \theta^2(\tau,z)/\eta^6(\tau)$ of weight -2 and index 1 in the same way as before:

$$\begin{aligned}\tilde{\phi}(\tau,z) =& \tilde{h}_0(\tau)\,\theta_3(2\tau,2z) + \tilde{h}_1(\tau)\,\theta_2(2\tau,2z)\\ \tilde{h}_0(\tau) =& \frac{\theta_2(2\tau)}{\eta^6} = 2 + 12q + 56q^2 + \dots\ ,\\ \tilde{h}_1(\tau) =& -\frac{\theta_3(2\tau)}{\eta^6} = -q^{-1/4}(1 + 8q + 39q^2 + \dots)\ .\end{aligned} \tag{42}$$

The corresponding weight $-5/2$ modular form in Kohnen's plus space is

$$\begin{aligned}\tilde{h}(\tau) =& \tilde{h}_0(4\tau) + \tilde{h}_1(4\tau) = -\frac{\theta_4(2\tau)}{[\eta(4\tau)]^6} = \frac{\theta_2^8(2\tau) - 4X_2^2(2\tau)}{4\theta_3(2\tau)\Delta_6}\\ =& \sum_{m\geq -1} \tilde{c}(m) q^m = -\left(\frac{1}{q} - 2 + 8q^3 - 12q^4 + 39q^7 + \dots\right)\end{aligned} \tag{43}$$

As in the previous case, $\tilde{h}(\tau)$ has a simple pole at the cusp at infinity and is regular at the other cusps $\tau = 0$ and $\tau = \frac{1}{2}$. The action of the raising operator D_τ on $\tilde{h}_i$ evaluates to

$$D_\tau \tilde{h}_i = \frac{5}{12}\hat{E}_2\,\tilde{h}_i - \frac{1}{24} h_i\ , \quad i = 0, 1, \tag{44}$$

where $\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2}$ is the almost holomorphic Eisenstein series of weight 2 and h_i is defined in (30). The zero-th Fourier coefficient term of $D_\tau \tilde{h}_0$ is $-5/(2\pi\tau_2)$, so the integral (41) is convergent, with no need for regularization. Using (33) and the fact that $D_\tau \tilde{h}_i$ is an eigenmode of $\Delta_{SL(2),-1/2}$ with eigenvalue $5/2$, one easily checks that $\tilde{\varphi}$ is an eigenmode of $\Delta_{Sp(4)}$ with eigenvalue 5, away from the separating degeneration divisor $v = 0$. In the limit $v \to 0$, the integrand becomes

$$\Gamma_{2,2}\left[\theta_3(2\tau)\,D_\tau\tilde{h}_0(\tau) + \theta_2(2\tau)\,D_\tau\tilde{h}_1(\tau)\right] = -\Gamma_{2,2} \overset{\tau_2\to\infty}{\sim} -\tau_2\ , \tag{45}$$

leading to a logarithmic divergence. Keeping v small but non zero, retaining the contributions from $m_i = n^i = 0, b = \pm 1$ and using $D_\tau \tilde{h}_1 \sim -\frac{1}{2}q^{-1/4}(1 - \frac{5}{2\pi\tau_2})$, we have

$$\tilde{\varphi}(\Omega) \sim \frac{1}{2}\int_1^\infty \frac{\mathrm{d}\tau_2}{\tau_2} e^{-\frac{\pi\tau_2|v|^2}{\rho_2\sigma_2 - v_2^2}}\left(1 - \frac{5}{2\pi\tau_2}\right) = \frac{1}{2}(1 + \frac{5}{2}z)\,\Gamma(0,\pi z) - \frac{5}{4\pi}e^{-\pi z}\ , \tag{46}$$

where $z = \frac{|v|^2}{\rho_2\sigma_2 - v_2^2}$. Thus, in the separating degeneration, we have, in agreement with (8),

$$\tilde{\varphi}(\Omega) = -\frac{1}{2}\left(1 + \frac{5|v|^2}{2(\rho_2\sigma_2 - v_2^2)}\right)\log|v|^2 - \log|2\pi\eta^2(\rho)\eta^2(\sigma)| + \mathcal{O}(|v|^2)\ , \tag{47}$$

up to terms vanishing analytically as $v \to 0$ (the last term on the right-hand side follows from (37)). The logarithmic singularity implies that

$$\left[\Delta_{Sp(4)} - 5\right]\,\tilde{\varphi} = -2\pi\,\det(\Omega_2)\,\delta^{(2)}(v)\ , \tag{48}$$

so $\tilde{\varphi}$ satisfies the same equation (2) as φ. In the remainder of this subsection we extract the asymptotics of $\tilde{\varphi}$ in the minimal and maximal non-separating degenerations, and find that they agree with the asymptotics of φ.

Maximal non-separating degeneration. The maximal degeneration $\Omega_2 \to \infty$ corresponds, in string theory parlance, to the limit where one of the circles in the torus T^2 becomes infinitely large (see footnote 1). In this limit, the lattice partition function $\Gamma_{3,2}^{\rm even|odd}(\Omega)$ factorizes into $\Gamma_{1,1}(r;\tau)\times\,\Gamma_{2,1}^{\rm even|odd}(\tilde\tau;\tau)$, where $r=\sqrt{\det\Omega_2}$ parametrizes the radius of the large circle, and $\tilde\tau=u_2+\mathrm{i}\sqrt{t/\rho_2}\equiv Y+\mathrm{i}R$ the radius R and Wilson line Y for the circle of finite size. It is useful to express the lattice partition functions $\Gamma_{1,1}$ and $\Gamma_{2,1}$ in the 'Lagrangian' representation, where modular invariance in τ is manifest,

$$\begin{aligned}\Gamma_{1,1}(r;\tau) =& r\sum_{(p,q)\in\mathbb{Z}^2} e^{-r^2|p+q\tau|^2/\tau_2}\\ \Gamma_{2,1}^{\rm even|odd}(\tilde\tau;\tau) =& R\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ b\in 2\mathbb{Z}|2\mathbb{Z}+1}} e^{-\pi R^2\frac{|m+n\tau|^2}{\tau_2}+2\mathrm{i}\pi n(m+n\tau)Y^2+\frac{\mathrm{i}\pi\tau}{2}b^2+2\mathrm{i}\pi(m+n\tau)bY}\ .\end{aligned}\tag{49}$$

In the limit $r\to\infty$, the $O(3,2,\mathbb{Z})=Sp(4,\mathbb{Z})$ symmetry is broken to $O(2,1,\mathbb{Z})=GL(2,\mathbb{Z})$, acting on the modulus $\tilde\tau\in\mathcal{H}_1$ by fractional linear transformations, along with the anti-holomorphic involution $\tilde\tau\mapsto-\bar{\tilde\tau}$. The leading term in this limit originates from the term $(p,q)=(0,0)$ in $\Gamma_{1,1}(r)$,

$$\varphi_L=-\frac{r}{2}\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\left[\Gamma_{2,1}^{\rm even}(\tilde\tau;\tau)\,D_\tau\tilde h_0(\tau)+\Gamma_{2,1}^{\rm odd}(\tilde\tau;\tau)\,D_\tau\tilde h_1(\tau)\right]\ .\tag{50}$$

To compute this integral, we decompose the sum over (m,n) in $\Gamma_{2,1}$ into orbits of $SL(2,\mathbb{Z})$, obtaining $\varphi_L=\varphi_L^{(0)}+\varphi_L^{(1)}$. The first term corresponds to the contribution of the zero orbit $(m,n)=(0,0)$,

$$\varphi_L^{(0)}=-\frac{rR}{2}\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\left[\theta_3(2\tau)\,D_\tau\tilde h_0(\tau)+\theta_2(2\tau)\,D_\tau\tilde h_1(\tau)\right]=\frac{\pi rR}{6}\ ,\tag{51}$$

since, as already noted in (45), the term in square bracket reduces to -1. The remaining orbits $(m,0)$ with $m\neq0$ contribute

$$\varphi_L^{(1)}=-\frac{rR}{2}\int_{\mathcal{S}}\frac{\mathrm{d}^2\tau}{\tau_2^2}\sum_{m\neq0}e^{-\pi R^2m^2/\tau_2}\left[\theta_3(2\tau,2mY)\,D_\tau\tilde h_0(\tau)+\theta_2(2\tau,2mY)\,D_\tau\tilde h_1(\tau)\right]\tag{52}$$

where $\mathcal{S}$ is the strip $[-1/2,1/2]\times\mathbb{R}^+$. The integral over τ_1 picks up the constant term in $D_\tau\tilde h_0(\tau)$ (corresponding to $b=0$) and the polar term in $D_\tau\tilde h_1(\tau)$ (corresponding to $b=\pm1$). Thus we have

$$\begin{aligned}\varphi_L^{(1)}=&-\frac{r\,R}{2}\int_0^\infty\frac{\mathrm{d}\tau_2}{\tau_2^2}\sum_{m\neq0}e^{-\pi R^2m^2/\tau_2}\\ &\times\left[\frac{5}{12}\left(1-\frac{3}{\pi\tau_2}\right)\left(2-e^{2\pi\mathrm{i}mY}-e^{-2\pi\mathrm{i}mY}\right)-\frac{1}{24}\left(20+2\,e^{2\pi\mathrm{i}mY}+2\,e^{-2\pi\mathrm{i}mY}\right)\right]\\ =&-\frac{r}{2}\sum_{m=1}^\infty\left[5\frac{\cos(2\pi mY)-1}{\pi^3R^3m^4}-\frac{2\cos(2\pi mY)}{\pi Rm^2}\right]\end{aligned}\tag{53}$$

Using the identity $\mathrm{Li}_k(e^{2\pi i x}) + (-1)^k \mathrm{Li}_k(e^{-2\pi i x}) = -\frac{(2\pi i)^k}{k!} B_k(x)$ for the polylogarithm, valid for k integer, $0 < \mathrm{Re}(x) < 1$, we arrive, in the region $-\frac{1}{2} < Y < \frac{1}{2}, R^2 + Y^2 > 1$, at

$$\varphi_L \equiv \varphi_L^{(0)} + \varphi_L^{(1)} = \frac{\pi r}{6} \left[R + 5 \frac{Y^2(|Y|-1)^2}{R^3} + \frac{1 - 6|Y| + 6Y^2}{R} \right] \tag{54}$$

Setting $\rho_2 = r/R, u_2 = Y, t = rR$, this reproduces the desired behavior (14) in the maximal separating degeneration limit ! The square bracket in (54) is recognized as the local modular function $\hat{A}(\tilde{\tau})$ in the two-loop supergravity computation [20, Eq. (3.8)].

The subleading terms in the maximal non-separating degeneration limit are obtained by restricting the sum $\Gamma_{1,1}(r)$ to the orbit representatives $(p, 0)$ with $p \neq 0$, and unfolding on the strip:

$$\tilde{\varphi} - \varphi_L = -\frac{r}{2} \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \sum_{p \neq 0} e^{-\pi r^2 p^2/\tau_2} \left[\Gamma_{2,1}^{\text{even}}(R, Y)\, D_\tau \tilde{h}_0(\tau) + \Gamma_{2,1}^{\text{odd}}(R, Y)\, D_\tau \tilde{h}_1(\tau) \right] \tag{55}$$

Up to terms of order e^{-r}, one can replace $\Gamma_{2,1}^{\text{even}} \to \sqrt{\tau_2}$, $\Gamma_{2,1}^{\text{odd}} \to 0$ and $D_\tau \tilde{h}_0(\tau)$ by its constant term $-5/(2\pi\tau_2)$, leading to the next-to-leading correction

$$\varphi_{NL} = \frac{15r}{12\pi} \int_0^\infty \frac{d\tau_2}{\tau_2^{5/2}} \sum_{p \neq 0} e^{-\pi r^2 p^2/\tau_2} = \frac{5\zeta(3)}{4\pi^2 r^2}, \tag{56}$$

consistently with (22). The exponentially suppressed contributions to $\varphi - \varphi_L - \varphi_{NL}$ will be obtained in the analysis of the minimal non-separating degeneration, to which we now turn.

Minimal non-separating degeneration. The limit $\sigma_2 \to \infty$ keeping other entries of Ω fixed corresponds, in string theory parlance, to the limit where the volume of the torus T^2 becomes infinite, keeping the complex structure ρ and holonomy v fixed (see footnote 1). The Siegel modular group Γ is now broken to the Jacobi subgroup Γ_J. Following [7, 22], the Fourier-Jacobi coefficients (i.e. the Fourier coefficients with respect to σ_1) can be extracted by applying the orbit method to the lattice partition function written in the 'Lagrangian' representation, obtained from (25) by Poisson resummation in m_1, m_2,

$$\Gamma_{3,2}^{\text{even}|\text{odd}}(\Omega; \tau) = t \sum_{\substack{A \in \mathbb{Z}^{2\times 2} \\ b \in 2\mathbb{Z}|2\mathbb{Z}+1}} e^{\frac{i\pi\tau}{2} b^2 - \pi G(A)}, \tag{57}$$

where

$$G(A) = \frac{t|\mathcal{A}|^2}{\rho_2\, \tau_2} + 2i\sigma \det A + \frac{1}{\rho_2} b \cdot (\bar{v}\, \mathcal{A} - v\, \tilde{\mathcal{A}}) + \frac{n_2}{\rho_2}(v^2\, \tilde{\mathcal{A}} - \bar{v}^2\, \mathcal{A}) - 2i \frac{v_2^2}{\rho_2^2}(n_1 + n_2\bar{\rho})\, \mathcal{A} \tag{58}$$

and

$$A = \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 1 & \rho \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \qquad \tilde{\mathcal{A}} = \begin{pmatrix} 1 & \bar{\rho} \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \tag{59}$$

The integer matrix A transforms linearly under $SL(2,\mathbb{Z})$, and belongs to one of three different types of orbits. The orbit $A=0$ produces, as in (51),

$$\varphi^{(0)} = -\frac{1}{2}t\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\left[\theta_3(2\tau)\,D_\tau\tilde{h}_0(\tau)+\theta_2(2\tau)\,D_\tau\tilde{h}_1(\tau)\right]=\frac{\pi t}{6}\,. \tag{60}$$

The degenerate orbits with $\mathrm{rk}A=1$ give instead, restricting to the orbit representatives $(n_1,n_2)=(0,0)$, $(m_1,m_2)\neq(0,0)$ and unfolding on the strip,

$$\begin{aligned}\varphi^{(1)} = &-\frac{1}{2}t\int_{\mathcal{S}}\frac{\mathrm{d}^2\tau}{\tau_2^2}\sum_{(m_1,m_2)\neq 0}e^{-\frac{\pi t|m_1+m_2\rho|^2}{\tau_2\rho_2}}\\ &\times\left[\theta_3(2\tau,2m_1u_2+2m_2u_1)\,D_\tau\tilde{h}_0+\theta_2(2\tau,2m_1u_2+2m_2u_1)\,D_\tau\tilde{h}_1\right]\,.\end{aligned} \tag{61}$$

As in (53), the integral over τ_1 picks up the constant term in $D_\tau\tilde{h}_0$ (corresponding to $b=0$) and the polar term in $D_\tau\tilde{h}_1$ (corresponding to $b=\pm1$), leading to

$$\begin{aligned}\varphi^{(1)} &= -\frac{t}{2}\sum_{(m_1,m_2)\neq 0}\left[5\frac{\rho_2^2(\cos[2\pi(m_1u_2+m_2u_1)]-1)}{\pi^3t^2|m_1+m_2\rho|^4}-\frac{2\rho_2\cos[2\pi(m_1u_2+m_2u_1)]}{\pi t|m_1+m_2\rho|^2}\right]\\ &=\frac{1}{2}\mathcal{D}_{1,1}(\rho,v)+t^{-1}\left[\frac{5}{16\pi^2\rho_2}\mathcal{D}_{2,2}(\rho,v)+\frac{5}{2\pi}E^\star(2;\rho)\right]\,.\end{aligned} \tag{62}$$

Combining (60) and (62), we reproduce the desired behavior (9) in the minimal degeneration limit, with φ_0 and φ_1 given in (21) and (16) !

For the non-degenerate orbits with $\det A\neq 0$, the integral can be unfolded on (a double cover of) the upper half-plane, at the expense of restricting the sum to $n_2=0\le m_1<n_1\neq 0$. Substituting the Fourier series of $\tilde{h}_0$ and $\tilde{h}_1$, the integral over τ_1 is Gaussian, while the integral over τ_2 is of Bessel type. After some algebra (see e.g. [22, A.2]), we arrive at

$$\varphi^{(2)} = \sum_{\substack{k>0,\ell\ge 0\\ b\in\mathbb{Z}}}\tilde{c}(4k\ell-b^2)\left[-\frac{5}{16\pi^2t\rho_2}D_{2,2}\left(x\right)+\frac{1}{2}(4k\ell-b^2)D_{1,1}\left(x\right)\right]\,, \tag{63}$$

where $x=e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}$, and $D_{a,b}(x)$ are the single-valued polylogarithms defined in (20),

$$D_{1,1}(x)=2\,\mathrm{Re}[\mathrm{Li}_1(x)]\,,\quad D_{2,2}(x)=-4\,\mathrm{Re}[\mathrm{Li}_3(x)-\log|x|\;\mathrm{Li}_2(x)]\,. \tag{64}$$

The formula (63) holds in the chamber where $k\sigma_2+\ell\rho_2+bv_2>0$ for all stated values of (k,ℓ,b). The sum converges absolutely in a neighborhood of the zero-dimensional cusp $\Omega=\mathrm{i}\infty$ by the same arguments as in [7, 9].

3.3. The Kawazumi-Zhang and Faltings invariants as Theta lifts. Using the results in §3.2, §3.2 and the facts summarized in the introduction, we see that $\hat{\varphi}=\varphi-\tilde{\varphi}$ is annihilated by $\Delta_{Sp(4)}-5$ and vanishes up to order $\mathcal{O}(1/t)$ in the non-separating degeneration limit $t\to\infty$, and up to order $\mathcal{O}(|v|^2\log|v|)$ near the separating divisor. On the truncated fundamental domain $\mathcal{F}_2^\Lambda=\mathcal{F}_2\cap\{t<\Lambda,|v|>1/\Lambda\}$, where $\mathcal{F}_2$ is the standard fundamental domain from [18], one has

$$\int_{\mathcal{F}_2^\Lambda}\hat{\varphi}^2(\star 1)=\frac{1}{5}\int_{\mathcal{F}_2^\Lambda}\hat{\varphi}\,\Delta_{Sp(4)}\hat{\varphi}=-\frac{1}{5}\int_{\mathcal{F}_2^\Lambda}\mathrm{d}\hat{\varphi}\star\mathrm{d}\hat{\varphi}+\frac{1}{5}\int_{\partial\mathcal{F}_2^\Lambda}\hat{\varphi}\star\mathrm{d}\hat{\varphi}\,, \tag{65}$$

where $\star$ denotes the Hodge star on $\mathcal{H}_2$, and $\partial\mathcal{F}_2^\Lambda$ the boundary of $\mathcal{F}_2^\Lambda$. By the above estimates, the boundary term vanishes in the limit $\Lambda\to\infty$, while the first term converges to a finite, non-positive value. Since the left-hand side is non-negative, it follows that $\hat\varphi$ must vanish. Thus, we have shown the

Theorem 1. *The Kawazumi-Zhang invariant $\varphi(\Omega)$ for compact genus-two Riemann surfaces admits the Theta lift representation*

$$\varphi(\Omega) = -\frac{1}{2}\int_{\mathcal{F}_1}\frac{\mathrm{d}^2\tau}{\tau_2^2}\,\left[\Gamma_{3,2}^{\rm even}(\Omega;\tau)\,D_\tau\tilde h_0(\tau)+\Gamma_{3,2}^{\rm odd}(\Omega;\tau)\,D_\tau\tilde h_1(\tau)\right]\ , \tag{66}$$

where $\Gamma_{3,2}^{\rm even|odd}(\Omega;\tau)$ are the Siegel-Narain theta series defined in (25)*, and $(\tilde h_0,\tilde h_1)$ is the weight $-\frac{5}{2}$ vector-valued modular form appearing in the theta series decomposition* (42) *of the weak Jacobi form $\theta^2(\tau,z)/\eta^6$ of weight 2 and index 1.*

Corollary 1. *$\varphi(\Omega)$ satisfies the following improved asymptotics: in the minimal non-separating degeneration $t\to+\infty$,*

$$\varphi(\Omega) = \frac{\pi}{6}t+\varphi_0+\frac{\varphi_1}{t}+\mathcal{O}(e^{-t})\ , \tag{67}$$

where φ_0 and φ_1 are defined in (16)*; in the maximal non-separating degeneration $L_i\to+\infty$,*

$$\varphi(\Omega)=\frac{\pi}{6}\left[L_1+L_2+L_3-\frac{5\,L_1L_2L_3}{L_1L_2+L_2L_3+L_3L_1}\right]+\frac{5\zeta(3)}{4\pi^2\det\Omega_2}+\mathcal{O}(e^{-L_i})\ ; \tag{68}$$

in the separating degeneration $v\to0$,

$$\varphi(\Omega) = -\frac{1}{2}\left(1+\frac{5|v|^2}{2(\rho_2\sigma_2-v_2^2)}\right)\,\log|v|^2-\log|2\pi\eta^2(\rho)\eta^2(\sigma)|+\mathcal{O}(|v|^2)\ . \tag{69}$$

Corollary 2. *$\varphi(\Omega)$ admits the Fourier expansion*

$$\begin{aligned}\varphi(\Omega) =&\frac{\pi}{6}(\rho_2+\sigma_2-|v_2|)-\frac{5\pi}{6}\frac{|v_2|(\rho_2-|v_2|)(\sigma_2-|v_2|)}{\det\Omega_2}+\frac{5\zeta(3)}{4\pi^2\det\Omega_2}\\ &-\frac{5}{16\pi^2\det\Omega_2}\sum_{(k,\ell,b)>0}\tilde c(4k\ell-b^2)\,D_{2,2}\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right)\\ &+\frac{1}{2}\sum_{(k,\ell,b)>0}(4k\ell-b^2)\,\tilde c(4k\ell-b^2)\,D_{1,1}\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right)\ ,\end{aligned} \tag{70}$$

where $(k,\ell,b)>0$ was defined below (39)*, and $D_{1,1}(x)$, $D_{2,2}(x)$ are given in* (64)*. The Fourier expansion is absolutely convergent in a neighborhood of the zero-dimensional cusp $\Omega=\mathrm{i}\infty$.*

In [11], a relation between the Kawazumi-Zhang invariant $\varphi(\Sigma)$, the Faltings invariant $\delta(\Sigma)$ and the discriminant $\Delta(\Sigma)$ for hyperelliptic compact Riemann surfaces Σ was obtained. At genus two, all compact Riemann surfaces are hyperelliptic, and the discriminant is proportional to the Igusa cusp form Ψ_{10}. Corollory 1.8 in [11] states

$$\varphi(\Omega) = -3\log||\Psi_{10}||(\Omega)-\frac{5}{2}\delta_F(\Omega)-40\log2\pi\ . \tag{71}$$

Using (24), (66) and (71), we obtain

Corollary 3. *The Faltings invariant admits the Theta lift representation*

(72)
$$\delta_F(\Omega) = \int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2} \left[\Gamma_{3,2}^{\rm even}(\Omega;\tau)\, \frac{2\hat E_2 \tilde h_0 + 7h_0}{24} + \Gamma_{3,2}^{\rm odd}(\Omega;\tau)\, \frac{2\hat E_2 \tilde h_1 + 7h_1}{24} - 6\,\tau_2 \right] + 6\log\left(\frac{4}{3\sqrt{3}} e^{1-\gamma_E}\right) - 10\log 2\pi\ .$$

The Fourier expansion of $\delta_F(\Omega)$ is easily obtained by combining (39), (70) and (71).

4. Miscellany

4.1. Numerical applications. The formulae (39) and (70) provide an efficient numerical procedure for evaluating the Faltings and Kawazumi-Zhang invariants to arbitrary precision. As an illustration, for the curve $y^2 + y = x^5$ considered in [8], with automorphism group $\mathbb{Z}_5 \times \mathbb{Z}_2$ and period matrix

(73)
$$\Omega = \begin{pmatrix} -\zeta_5^4 & \zeta_5^2 + 1 \\ \zeta_5^2 + 1 & \zeta_5^2 - \zeta_5^3 \end{pmatrix} ,$$

we find, truncating the sum at $k, \ell, |b| \le 15$,

(74) $\varphi = 0.538011176205005048 61\dots\ ,\quad \delta_F = -16.6790574451477760445\dots\ ,$

where all displayed digits appear to be stable upon varying the truncation. This is consistent with the value $\delta_F = -16,679\dots$ which follows from the numerical computations in [8, §4.5].

For another example, consider the curve $y^2 = x^6 - 1$, with automorphism group $D_6 \times \mathbb{Z}_2$ and period matrix

(75)
$$\Omega = \begin{pmatrix} \frac{2\mathrm{i}}{\sqrt{3}} & \frac{\mathrm{i}}{\sqrt{3}} \\ \frac{\mathrm{i}}{\sqrt{3}} & \frac{2\mathrm{i}}{\sqrt{3}} \end{pmatrix} .$$

Using the same truncation, we find

(76) $\varphi = 0.59291015631443383207\dots\ ,\quad \delta_F = -16.3412295821338262636\dots$

Finally, consider the Burnside curve $y^2 = x(x^4 - 1)$, with automorphism group $S_4 \times \mathbb{Z}_2$ and period matrix

(77)
$$\Omega = \begin{pmatrix} -\frac12 + \frac{\mathrm{i}}{\sqrt{2}} & \frac12 \\ \frac12 & -\frac12 + \frac{\mathrm{i}}{\sqrt{2}} \end{pmatrix} .$$

Using the same truncation, we find

(78) $\varphi = 0.51986038541995901150\dots\ ,\quad \delta_F = -16.8264632650009721134\dots$

4.2. Relation to Gromov-Witten invariants. We note that the Fourier expansion (70) is similar to [22, Eq. (A.44)], where modular integrals of the form $\int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2} \Gamma_{8t+2,2} \hat{E}_2\, F(\tau)$ were considered. Here, $\Gamma_{8t+2,2}$ is a partition of an even self-dual lattice of signature $(8t+2,2)$ and $F(\tau)$ is a weakly holomorphic modular form of weight $-4t-2$. In this context, the analogue of the coefficients $\tilde{c}(4k\ell - b^2)$ were identified as the BPS invariants (also known as Gopakumar-Vafa invariants, and closely related to Gromov-Witten invariants) counting rational curves in a suitable K3-fibered Calabi-Yau threefold with $h_{1,1} = 8t+3$. It is therefore natural to ask if the coefficients $\tilde{c}(4k\ell - b^2)$ in the Fourier expansion of the Kawazumi-Zhang invariant count rational curves in a suitable Calabi-Yau threefold with $h_{1,1} = 4$. Two examples of threefolds with $h_{1,1} = 4$ were studied in [6,24] (see also [27]). For the example $X(1,1,2,6,10)_{-372}$ in [6,27], the rational curves are counted by the weight -2 Jacobi form $-(7E_4E_{6,1} + 5E_6E_{4,1})/(6\,\eta^{24})$. For the example $X(2,2,3,3,10)_{-132}$ in [24], they are instead counted by the weight -2 Jacobi form $-2E_4E_{6,1}/\eta^{24}$. The Jacobi form relevant for the Kawazumi-Zhang invariant is proportional to the difference of these two, $\theta^2/\eta^6 = (E_4E_{6,1} - E_6E_{4,1})/(144\,\eta^{24})$. It is unclear to the author whether the fact that it is a weak Jacobi form (i.e. has $h(\tau) = \mathcal{O}(1/q)$ rather than $\mathcal{O}(1/q^4)$ as in the cases studied in [6,24,27]) disqualifies it from counting rational curves.

4.3. Holomorphic prepotential. It is known from [22] that modular integrals of the form

$$\mathcal{I} = \int_{\mathcal{F}_1} \frac{\mathrm{d}^2\tau}{\tau_2^2} \left[\Gamma_{3,2}^{\mathrm{even}}(\Omega;\tau)\, H_0(\tau) + \Gamma_{3,2}^{\mathrm{odd}}(\Omega;\tau)\, H_1(\tau) - c(0,0)\,\tau_2 \right] , \tag{79}$$

where $h(\tau) = h_0(4\tau) + h_1(4\tau) = \sum_{m \geq -\kappa, 0 \leq \ell \leq 1} c(m,\ell) q^m/\tau_2^\ell$ is an almost weakly holomorphic modular form of weight $-1/2$ and depth 1 under $\Gamma_0(4)$ in Kohnen's plus space, can be expressed as

$$\mathcal{I} = \mathrm{Re}\, F_0 + \mathrm{Re}\, \square_{-2} F_1 - c(0,0) \log\det \Omega_2 , \tag{80}$$

where F_0 and F_1 are holomorphic functions of Ω. Here, $\square_w$ is the Siegel-Maass raising operator

$$\square_w = -\frac{1}{\pi^2} \left[\partial_\rho \partial_\sigma - \tfrac14 \partial_v^2 + \frac{\mathrm{i}(1-2w)}{4(\rho_2\,\sigma_2 - v_2^2)} \left(\frac{w}{2\mathrm{i}} + \sigma_2 \partial_\sigma + \rho_2 \partial_\rho + v_2 \partial_v \right) \right] , \tag{81}$$

which maps Siegel modular forms of weight w to modular forms of weight $w+2$. F_0 is the logarithm of a holomorphic Siegel modular form of weight $-2c(0,0)$. F_1, known as the holomorphic prepotential, is ambiguous modulo elements in the kernel $\mathcal{K}$ of the operator $\mathrm{Re}(\square_{-2})$. The latter includes quadratic polynomials in (ρ, σ, v) with imaginary coefficients, as well as cubic polynomials of the form $(\rho\sigma - v^2)(\alpha\rho + \beta\sigma)$ where α, β are imaginary. Since the integral $\mathcal{I}$ is a Siegel modular function, F_1 must transform under $\gamma \in Sp(4,\mathbb{Z})$ as

$$F_1|_{-2}\gamma\,(\Omega) = F_1(\Omega) + P_\gamma(\Omega) , \tag{82}$$

where $P_\gamma(\Omega)$ is an element in $\mathcal{K}$. Thus, F_1 is a mock-type holomorphic Siegel modular form of weight -2. For the modular integral (24), the modular form $H(\tau) = -\frac14 h(\tau)$ is weakly holomorphic therefore F_1 vanishes, while $F_0 = \log \Psi_{10}$ (up to an additive constant). For the modular integral (41), $H(\tau) = -\frac18 D_\tau \tilde{h}(\tau)$ is the modular derivative of a weakly holomorphic form, therefore F_0 vanishes [2,4]. Since, by Theorem 1, $\varphi(\Omega)$ is equal to $\mathcal{I}$ for this choice of $H(\tau)$, we have the

Corollary 4. *The Kawazumi-Zhang invariant $\varphi(\Omega)$ is equal to the real part of the action of the Siegel-Maass raising operator $\Box_{-2}$ on the 'holomorphic prepotential' $F_1(\Omega)$,*

$$\varphi = \mathrm{Re}\,(\Box_{-2} F_1) \tag{83}$$

where

$$F_1(\Omega) = \sum_{(k,\ell,b)>0} \tilde c(4k\ell - b^2)\,\mathrm{Li}_3\left(e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}\right) - \frac{\mathrm{i}\pi^3}{3}\rho\sigma(\rho+\sigma-2v) + \zeta(3)\ . \tag{84}$$

Proof: Using [26, A.32]

$$\Box_{-2n}^{n}\mathrm{Li}_{2n+1}(x) = \sum_{r=0}^{n} \frac{n!(n+r-w)!}{r!(n-r)!\,(n-w)!}\,\frac{2^{2n-2r}\,(k\ell-b^2)^{n-r}}{(\pi\det\Omega_2)^r}\,L_{(r)}\left(\frac{\log x}{2\pi\mathrm{i}}\right) \tag{85}$$

with $n=2$, where $x = e^{2\pi\mathrm{i}(k\sigma+\ell\rho+bv)}$ and

$$L_{(r)}(z) = \sum_{m=0}^{r} \frac{(r+m)!}{m!(r-m)!(4\pi)^m}[\mathrm{Im}\,z]^{r-m}\,\mathrm{Li}_{r+m+1}(e^{2\pi iz}) \tag{86}$$

is related to the Bloch-Wigner-Ramakrishnan polylogarithm $D_{a,b}$ in (20) via [2]

$$D_{r+1,r+1}(x) = 2\,\mathrm{Re}\left[\frac{(-4\pi)^r}{r!}\,L_{(r)}\left(\frac{\log x}{2\pi\mathrm{i}}\right)\right]\ , \tag{87}$$

one easily checks that the action of $\mathrm{Re}(\Box_{-2})$ on the first term of (84) produces the last two lines in (70). The action of the same on the polynomial terms in (84) produces the first line in (70). $\Box$

Remark. More generally, integrals of the form (79), where $H(\tau)$ is an almost weakly holomorphic modular form of weight $-1/2$ and depth n, can be expressed as

$$\mathcal{I} = \sum_{r=0}^{n} \mathrm{Re}\,\Box_{-2r}F_r - c(0,0)\log\det\Omega_2\ , \tag{88}$$

where F_r are holomorphic functions of Ω known as generalized prepotentials, which transform as mock-type Siegel modular forms of weight $-2r$ [2,4,26]. When $H(\tau)$ is obtained by acting r times with the raising operator D_τ on a holomorphic modular form $h(\tau)$ of weight $-2r-\frac{1}{2}$, then all F_r vanish except F_n [2,4].

4.4. Quartic differential equation. Observe that the Narain partition function satisfies, in addition to (33),

$$\left[\underline{\Box}_2\,\Box_0 - \frac{1}{16\pi^4}\Delta_{SL(2),1/2}\left(\Delta_{SL(2),1/2}-\frac{1}{2}\right)\right]\Gamma_{3,2}^{\mathrm{even}|\mathrm{odd}} = 0\ , \tag{89}$$

where $\underline{\Box}_w$ is the Siegel-Mass lowering operator (formally independent of w)

$$\underline{\Box}_w = -\pi^2(\rho_2\sigma_2 - v_2^2)^2\left[\partial_{\bar\rho}\partial_{\bar\sigma} - \tfrac{1}{4}\partial_{\bar v}^2 - \frac{\mathrm{i}}{4(\rho_2\,\sigma_2 - v_2^2)}\left(\sigma_2\partial_{\bar\sigma} + \rho_2\partial_{\bar\rho} + v_2\partial_{\bar v}\right)\right]\ , \tag{90}$$

which maps Siegel modular forms of weight w to Siegel modular forms of weight $w-2$. By integration by parts, we conclude that

Corollary 5. *$\varphi(\Omega)$ satisfies the quartic differential equation (away from the separating degeneration)*

$$\left(\underline{\square}_2\,\square_0-\frac{15}{32}\right)\varphi=0\,. \tag{91}$$

It would be interesting to understand the fate of the differential equations (2) and (91) at higher genus.

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Boris Pioline, CERN, Theory Division, Case C01600, CH-1211 Geneva 23
On leave from: LPTHE, CNRS UMR 7589 and Université Pierre et Marie Curie,
4 place Jussieu, F-75005, Paris, France.
e-mail: boris.pioline@cern.ch